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DETERMINING INTERMEDIATE ORIGIN-DESTINATION MATRICES FOR THE ANALYSIS OF COMPOSITE MODE TRIPS

BY

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ABSTRACT

We consider in this paper the problem of determining intermediate origin-destination matrices for composite mode trips that involve a trip by private car to a parking facility and the continuation of the trip to the destination either by walking or by a transit mode. The intermediate origin-destination matrices relate to each component of the composite mode trip: a matrix from the trip origins to intermediate destinations which are parking lots and a matrix from the parking lots to the final destinations. The approach that we propose to solve this problem is to modify the entropy based trip distribution models to consider inequality constraints related to parking lot capacities. Such models may be easily calibrated by using well known calibration methods or generalization of these methods and may be easily solved by applying a primal feasible direction method of nonlinear programming.

1. INTRODUCTION

We consider in this paper the problem of determining intermediate origin destination (O/D) matrices for composite mode trips that involve a trip by private car to a parking facility and the continuation of the trip to the destination either by walking or by a transit mode. The intermediate O/D matrices relate to each component of the composite mode trip: an O/D matrix from the trip origins to the intermediate destinations which are parking lots and an O/D matrix from the parking lots to the final destinations.

Typical examples are trips taken to a CBD destination by private car to a parking lot and the continuation of this trip to the final destination on foot, or so-called 'park-and-ride' trips taken by private car to a parking lot near a train station, then by train to a station near the final destination. The interest in this problem stems from the need to predict changes in traffic flows that would result from changes in parking policy, such as decreases or increases in parking lot capacities, additional parking facilities and changes in pricing of parking facilities. The modelling approaches proposed and used to date, which we review in detail below, have not succeeded in handling successfully the effects of changes in parking capacities. The models which we develop in this paper, which are based on entropy maximization methodology take explicitly into account parking lot capacities, as well as other relevant factors, which we believe to be novel and of potential use in practice.

Three types of approaches have been proposed previously for modelling the attraction of parking lots: disaggregate demand models for parking, linear programming based policy lot allocation models and trip distribution type models.

The disaggregate demand models (see, for example, Gillen, 1975 and Westin and Gillen, 1976) attempt to predict individual choice for parking lot as a function of independent explanatory variables which characterize the trip maker, such as income, sex, profession, and the parking lot location, such as proximity to the final destination, cost, etc. So far, these models have not considered the capacities of the parking lots and their indiscriminate use may result in demand predictions which exceed these capacities.

Linear programming based parking lot allocation models use zonal aggregated data and determine the choice of parking lots as a result of the minimization of an aggregate travel cost. These models take into account parking lot capacities explicitly; however the objective function minimized is normative and may not be a suitable representation of trip makers behavior (see, for example, Austin, 1973 and Whitlock, 1973).

Finally, the gravity type distribution model has been suggested for handling versions of this problem (see, for example, Kanafani, 1972). The drawback of this approach is that parking lot capacities must always be filled, since one is limited to the traditional double equality constraints. Thus, one does not allow here for the possibility of underutilizing a parking lot. This is a serious limitation since one of the aims is to determine those lots which are fully utilized and those which are not.

In summary, the previous approaches have failed to model in a satisfactory way the interaction between the demand and supply for parking in a spatial context. The approach that we propose is to modify the entropy based trip distribution models to consider inequality constraints. Although the trip maker's behaviour is modelled in a relatively crude way, we believe that these models are good compromises that allow the interaction between demand and supply.

The paper is organized in the following way: In section 2 we formulate the mathematical models that we propose as minimization problems and study the properties of the solutions by using the Kuhn-Tucker conditions; the calibration of these models is discussed in section 3, where we also show that Evans' (1971) results for the calibration of trip distribution models with two constraints can be extended to such models with three constraints; the solution algorithms that we propose are presented in section 4 and alternative modelling approaches as well as modifications required to handle 'park-and-ride' situations are presented in section 5.

2. MATHEMATICAL FORMULATION OF THE MODELS

The first model that we propose is stated under the assumption that an origin-destination matrix of person trips, g_{ij}, between origins i, $i \in I$, and final destinations j, $j \in J$, is known. Each trip is taken first by car to a parking lot k, $k \in K$, and on foot to the final destination.

The average car occupancy factors γ_k , may vary for each parking lot. The in-vehicle travel cost per person, measured in generalized cost units, from origin i to parking lot k is c_{ik} and c_{kj} is the walking cost between parking lot k and the final destination j. Let $f_{ij,k}$ be the number of person trips between i and j that use parking lot k . We require $f_{ij,k}$ to be the solution to the minimization problem

$$\text{Min} \sum_{(i,j)} \sum_{k} f_{ij,k} \{\ell n\, f_{ij,k} + \theta\, (c_{ik} + c_{kj})\} \qquad (1)$$

$$\text{subject to} \quad \sum_{k} f_{ij,k} = g_{ij} , \quad \text{for all possible pairs } (ij) \qquad (2)$$

$$\sum_{(i,j)} f_{ij,k} / \gamma_k \leq T_k , \quad \text{for all } k \qquad (3)$$

$$f_{ij,k} \geq 0 , \quad \text{for all } (ij,k) \qquad (4)$$

where θ is a parameter to be calibrated from data and T_k is the parking capacity of lot k . The structure of this model is more transparent if we simplify the notation by indexing with ℓ each pair (i,j) , that is $L = \{\ell \mid (ij) , i \in I , j \in J\}$.

Consequently we have that $f_{\ell,k}$ are the solution to the problem:

$$\text{Min} \sum_{\ell} \sum_{k} f_{\ell,k} \{\ell n\, f_{\ell,k} + \theta\, c_{\ell,k}\} \qquad (5)$$

$$\text{subject to} \quad \sum_{k} f_{\ell,k} = g_{\ell} , \quad \text{all } \ell \qquad (6)$$

$$\sum_{\ell} f_{\ell,k} \leq \gamma_k \cdot T_k \qquad \text{all } k \tag{7}$$

$$f_{\ell,k} \geq 0 \tag{8}$$

where we have replaced $c_{ik} + c_{kj}$ by $c_{\ell,k}$.

The model (5) - (8) is nearly the entropy maximization formulation of a doubly constrained trip distribution model, except for (7). If the T_k are very large and therefore these latter constraints are not binding, (5), (6), (8) is a singly constrained gravity model. However, constraints (7) are essential in achieving our aim.

We recall that the objective function is convex in $f_{\ell,k}$ and we set $f_{\ell,k} \ln f_{\ell,k} = 0$ at $f_{\ell,k} = 0$. The Kuhn-Tucker conditions (see, for example, Zangwill, 1969) are necessary and sufficient conditions for the optimal solution $f^*_{\ell,k}$. The conditions, after a few suitable modifitations, are:

$$f^*_{\ell,k} = \exp(-v_k - z_\ell - 1) \cdot \exp(-\theta\, c_{\ell,k}) \tag{9}$$

where v_k and z_ℓ are the multipliers associated with parking lot k and origin-destination pair ℓ. If we let $r_k = \exp(-v_k)$ and $s_\ell = \exp(-z_\ell - 1)$ we obtain

$$f^*_{\ell,k} = r_k \cdot s_\ell \cdot \exp(-\theta\, c_{\ell,k}) \tag{10}$$

The proportion of trips from origin-destination ℓ that use parking lot k is given by

$$\frac{f^*_{\ell,k}}{\sum_k f^*_{\ell,k}} = \frac{\exp(-v_k) \cdot \exp(-\theta\ c_{\ell,k})}{\sum_k \exp(-v_k) \cdot \exp(-\theta\ c_{\ell,k})} \qquad (11)$$

In addition we have the complementarity conditions

$$v_k > 0 \quad \Rightarrow \gamma_k T_k - \sum_\ell f^*_{\ell,k} = 0 \qquad (12)$$

$$\gamma_k T_k - \sum_\ell f^*_{\ell,k} > 0 \quad \Rightarrow v_k = 0$$

If no parking lot were used at full capacity, then $v_k = 0$ for all $k \in K$ and the parking allocation equation (11) reduces to a logit model

$$\frac{f^*_{\ell,k}}{\sum_k f^*_{\ell,k}} = \frac{\exp(-\theta\ c_{\ell,k})}{\sum_k \exp(-\theta\ c_{\ell,k})} \quad ; \qquad (13)$$

otherwise the Langrange multipliers measure the attractiveness of lot k, and the term $\exp(-v_k)$ 'weights', as it were, the simple logit allocation. This is precisely the effect we are seeking to model, in contrast to the simple parking lot choice models.

The second model that we propose is more general, since we assume known, only the number of person trips originating at i, O_i and the number of

person trips destined for j, D_j , instead of an origin-destination matrix g_{ij} . The travel cost c_{ikj} is $c_{ik} + c_{kj}$ as before, and we require f_{ikj} , the number of trips that originate at i, use parking lot k and are destined for j, to be the solution to

$$\text{Min} \sum_i \sum_k \sum_j f_{ikj} \{\ell n \ f_{ikj} + \theta \cdot c_{ikj}\} \tag{14}$$

subject to

$$\sum_k \sum_j f_{ikj} = O_i \qquad \text{all } i \tag{15}$$

$$\sum_i \sum_k f_{ikj} = D_j \qquad \text{all } j \tag{16}$$

$$\sum_i \sum_j f_{ikj} \leq \gamma_k \cdot T_k \qquad \text{all } k \tag{17}$$

$$f_{ikj} \geq 0 \tag{18}$$

The interpretation of the constraints (15) - (17) is evident: in addition to the usual double constraints it is also required that the parking lot capacities are not exceeded.

The Kuhn-Tucker conditions for this model are:

$$f^*_{ikj} = \exp(-v_k) \cdot \exp(-\alpha_i - \beta_j - 1) \cdot \exp(-\theta \ c_{ikj}) \tag{19}$$

or

$$f^*_{ikj} = u_k \cdot r_i \cdot s_j \ \exp(-\theta \ c_{ikj}) \tag{20}$$

with $u_k = \exp(-v_k)$ $\qquad s_j = \exp(-\beta_j - 1)$

$r_i = \exp(-\alpha_i)$

The parking lot share for a given origin destination pair is analogous to (11), that is

$$\frac{f^*_{ikj}}{\sum_k f^*_{ikj}} = \frac{\exp(-v_k) \cdot \exp(-\theta\ c_{ikj})}{\sum_k \exp(-v_k) \cdot \exp(-\theta\ c_{ikj})} \tag{21}$$

and may be interpreted in a similar manner, since the complementarity condition (12) hold for this model as well.

In this model, an origin destination matrix g_{ij} is determined as $\sum_k f_{ikj}$ and the intermediate origin destination matrices are $\sum_j f_{ikj}$ and $\sum_i f_{ikj}$.

Both models have the property that we require, that is, explicit consideration of parking lot capacities. However, unlike conventional distribution models with equality constraints, the 'balancing' algorithm may not be used to obtain the explicit solutions (10) and (20) due to the presence of the inequality constraints (3), (17). Thus we must devise a different solution procedure. Also, we must show how to calibrate the parameter θ for each of these models.

3. CALIBRATION OF THE MODELS

For given values of θ, the two models that we propose are minimization problems with a convex objective function and linear constraints, which may be solved by adapting a suitable algorithm, such as the Frank and Wolfe (1956) algorithm. Thus, we propose to first calibrate θ on observed data related to parking lot usage, which permits to determine the utilization of each parking lot considered in the model. Thus $\hat{T}_k$ is the actual usage of parking lot k. For calibration purposes we replace (7) and (17) by equality constraints with the right hand side equal to $\gamma_k \hat{T}_k$. Then, we solve the original model with the value of θ so calibrated.

The first model that we proposed becomes

$$f_{\ell,k} = r_k \cdot s_\ell \cdot \exp(-\theta\ c_{\ell,k})$$

$$\text{subject to } \sum_k f_{\ell,k} = g_\ell \quad , \quad \text{all } \ell$$

$$\sum_\ell f_{\ell,k} = \gamma_k \hat{T}_k \quad , \quad \text{all } k$$

which is a classical doubly constrained trip distribution model (see, for example, Wilson, 1970). θ may be calibrated by using Evans' (1971) result that the maximum likelihood estimate of θ is the value chosen so that the mean trip cost given by the model $C(\theta)$, and stated below, equals the mean trip cost of the data $\hat{C}$.

Let $$C(\theta) = \frac{1}{G} \sum_k \sum_\ell c_{\ell,k} \cdot f_{\ell,k}(\theta)$$

with $$G = \sum_k \sum_\ell f_{\ell,k} = \sum_\ell g_\ell$$

be the mean trip cost given by the model. Evans also showed that $C(\theta)$ is a strictly decreasing function of θ, thus, an efficient search procedure may be devised for finding the optimal θ. This property is the basis of many calibration algorithms, such as those surveyed by Williams (1976). The approach taken in these methods is to choose an initial estimate of θ, compute $C(\theta) > \hat{C}$ then θ is decreased until equality is achieved; if $C(\theta) < \hat{C}$, then θ is increased until equality is achieved.

The calibration of θ for the second model is a more complex task. When (17) is replaced with equality constraints we obtain that f_{ikj} must satisfy

$$f_{ikj} = u_k \cdot r_i \cdot s_j \cdot \exp(-\theta\, c_{ikj}) \qquad (14')$$

$$\sum_k \sum_j f_{ikj} = O_i, \qquad \text{all } i \qquad (15)$$

$$\sum_i \sum_k f_{ikj} = D_j, \qquad \text{all } j \qquad (16)$$

$$\sum_i \sum_j f_{ikj} = \gamma_k \hat{T}_k, \qquad \text{all } k \qquad (17')$$

Before suggesting a calibration procedure we prove extensions of Evans' (1971) results regarding the doubly constrained trip distribution model:

. the maximum likelihood estimate of θ for the triply constrained model is that which equalizes the mean trip cost predicted by the model, $C(\theta)$ and the observed mean trip cost $\bar{C}$.

. the mean trip cost predicted by the model,

$$C(\theta) = \frac{1}{F} \sum_i \sum_k \sum_j c_{ikj} \cdot f_{ikj}(\theta) ,$$

with $F = \sum_i O_i = \sum_j D_j = \sum_i \sum_k \sum_j f_{ikj}(\theta)$, is a strictly decreasing function of θ (except for a trivial case to be mentioned).

In order to derive a maximum likelihood estimate of θ, we first note that u_k, r_i and s_j in (14') are uniquely determined if the dual variables γ_k, α_i and β_j are uniquely determined. This is due to the strict monotonicity of the exponential function. This is indeed the case: the function (14) being convex, there is a unique minimum to (14), (15), (16), (17') and therefore unique values for the dual variables, given a value of θ, (14') can therefore be restated as:

$$f_{ikj} = u_k(\theta)\, r_i(\theta)\, s_j(\theta) \exp(-\theta\, c_{ijk}) \qquad (14'')$$

We introduce new variables t_{ijk} :

$$t_{ikj} = \frac{f_{ikj}}{F}$$

with $$\sum_i \sum_j \sum_k t_{ikj} = 1 \qquad (22)$$

As done by Evans (1971) we assume that we have carried out a traffic survey and have observed a total number of n random trips, of which n_{ikj} are from origin zone i to destination zone j through parking k, so that:

$$n = \sum_i \sum_j \sum_k n_{ikj}$$

The probability that a random trip is from i to j through k is assumed given by

$$t_{ikj} = u_k(\theta)\, r_i(\theta)\, s_j(\theta) \exp(-\theta\, c_{ikj})$$

if u_k is appropriately redefined. The joint probability distribution of the sample frequencies, the n_{ikj}'s, is assumed multinomial. The likelihood of the observations is (see for instance Feller (1976), p. 168)

$$K \prod_i \prod_j \prod_k (t_{ikj})^{n_{ikj}}$$

with $$K = \frac{n!}{\prod_i \prod_j \prod_k (n_{ikj}!)}$$

Therefore:

$$\ell n\, L = \sum_i \sum_j \sum_k n_{ikj} (\ell n\, u_k + \ell n\, r_i + \ell n\, s_j - \theta\, c_{ikj}) + \ell n\, K . \qquad (23)$$

The maximum likelihood estimates of the u's, r's, s's and θ are obtained by maximizing $\ell n\, L$, subject to the constraint:

$$\sum_i \sum_j \sum_k t_{ikj} = \sum_i \sum_j \sum_k u_k r_i s_j \exp(-\theta c_{ikj}) = 1$$

Let λ be the associated Lagrange multiplier. The Lagrangian is:

$$\mathcal{L} = \sum_i \sum_j \sum_k n_{ikj} (\ln u_k + \ln r_i + \ln s_j - \theta c_{ikj})$$

$$- \lambda \left[\sum_i \sum_j \sum_k u_k r_i s_j \exp(-\theta c_{ikj}) - 1\right] \qquad (24)$$

Differentiating $\mathcal{L}$ with respect to u_k , r_i , s_j and θ yields:

$$\frac{\partial \mathcal{L}}{\partial u_k} = \left(\sum_i \sum_j n_{ikj}\right)\left(\frac{1}{u_k}\right) - \lambda \sum_i \sum_j r_i s_j \exp(-\theta c_{ikj}) = 0$$

$$\sum_i \sum_j n_{ikj} = \lambda \sum_i \sum_j u_k r_i s_j \exp(-\theta c_{ikj}), \text{ all } k$$

Summing over k, we obtain :

$$\lambda = n$$

and therefore

$$\frac{1}{n} \sum_i \sum_j n_{ikj} = \sum_i \sum_j u_k r_i s_j \exp(-\theta c_{ikj}) \qquad (25)$$

Similarly, the equations

$$\frac{\partial \mathcal{L}}{\partial r_i} = 0 \quad \text{all i} \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial s_j} = 0 \quad \text{all j}$$

entail:

$$\frac{1}{n} \sum_k \sum_j n_{ikj} = \sum_k \sum_j u_k r_i s_j \exp(-\theta c_{ikj}) \qquad (26)$$

$$\frac{1}{n} \sum_i \sum_k n_{ikj} = \sum_i \sum_k u_k r_i s_j \exp(-\theta c_{ikj}) \qquad (27)$$

Finally, $\frac{\partial \mathcal{L}}{\partial \theta} = 0$, entails,

$$\sum_i \sum_j \sum_k (-n_{ikj} c_{ikj}) - \lambda \sum_i \sum_j \sum_k (-c_{ikj}) t_{ikj} = 0$$

and therefore,

$$\frac{1}{n} \sum_i \sum_j \sum_k n_{ikj} c_{ikj} = \sum_i \sum_j \sum_k c_{ikj} u_k r_i s_j \exp(-\theta c_{ikj}) \qquad (28)$$

Equation (28) states that the 'observed' mean trip cost is equal to the mean trip cost given by the model □.

We now prove the second result, namely that the mean trip cost $C(\theta)$ strictly decreases with θ, for all θ, except in a trivial case.

The model is given by the equations:

$$t_{ikj}(\theta) = u_k(\theta) r_i(\theta) s_j(\theta) \exp(-\theta c_{ikj}) \qquad (29)$$

$$\sum_j \sum_k t_{ikj}(\theta) = a_i \quad , \text{ all } i \qquad (30)$$

$$\sum_i \sum_k t_{ikj}(\theta) = b_j \quad , \text{ all } j \qquad (31)$$

$$\sum_i \sum_j t_{ikj}(\theta) = \pi_k \quad , \text{ all } k \qquad (32)$$

with $$a_i = \frac{O_i}{F}$$

$$b_j = \frac{D_j}{F}$$

$$\pi_k = \frac{\gamma_k T_k}{F}$$

The derivative of $C(\theta) = \sum_i \sum_j \sum_k c_{ikj} t_{ikj}(\theta)$ with respect to θ is

$$\frac{dC}{d\theta} = \sum_i \sum_j \sum_k c_{ikj} \frac{dt_{ikj}}{d\theta} \tag{33}$$

Equation (29) implies:

$$\frac{1}{t_{ikj}} \frac{dt_{ikj}}{d\theta} = \frac{1}{u_k} \frac{du_k}{d\theta} + \frac{1}{r_i} \frac{dr_i}{d\theta} + \frac{1}{s_j} \frac{ds_j}{d\theta} - c_{ikj} \tag{34}$$

We introduce the following definitions:

$$w_k(\theta) = \frac{1}{u_k} \frac{du_k}{d\theta} \quad , \text{ all } k \tag{35}$$

$$p_i(\theta) = \frac{1}{r_i} \frac{dr_i}{d\theta} \quad , \text{ all } i \tag{36}$$

$$q_j(\theta) = \frac{1}{s_j} \frac{ds_j}{d\theta} \quad , \text{ all } j \tag{37}$$

(34) is restated as:

$$\frac{dt_{ikj}}{d\theta} = [w_k(\theta) + p_i(\theta) + q_j(\theta) - c_{ikj}] t_{ikj}(\theta) \tag{38}$$

and (33) is restated as:

$$\frac{dC}{d\theta} = \sum_i \sum_j \sum_k [w_k(\theta) + p_i(\theta) + q_j(\theta) - c_{ikj}]\, t_{ikj}(\theta)\, c_{ikj} \qquad (39)$$

Taking the derivatives of (30), (31) and (32) with respect to θ yields:

$$\sum_j \sum_k [w_k(\theta) + p_i(\theta) + q_j(\theta) - c_{ikj}]\, t_{ikj}(\theta) = 0 \qquad (40)$$

$$\sum_i \sum_k [w_k(\theta) + p_i(\theta) + q_j(\theta) - c_{ikj}]\, t_{ikj}(\theta) = 0 \qquad (41)$$

$$\sum_i \sum_j [w_k(\theta) + p_i(\theta) + q_j(\theta) - c_{ikj}]\, t_{ikj}(\theta) = 0 \qquad (42)$$

(40), (41) and (42) yield:

$$\sum_i \sum_j \sum_k p_i\, [w_k + p_i + q_j - c_{ikj}]\, t_{ikj} = 0 \qquad (43)$$

$$\sum_i \sum_j \sum_k q_j\, [w_k + p_i + q_j - c_{ikj}]\, t_{ikj} = 0 \qquad (44)$$

$$\sum_i \sum_j \sum_k w_k\, [w_k + p_i + q_j - c_{ikj}]\, t_{ikj} = 0 \qquad (45)$$

Substracting the left hand sides of (43), (44) and (45) from (39) gives:

$$\frac{dC}{d\theta} = -\sum_i \sum_j \sum_k [w_k + p_i + q_j - c_{ikj}]^2\, t_{ikj}$$

Therefore, unless $c_{ikj} = w_k + p_i + q_j$ (i.e. $[c_{ikj}]$ is a trivial matrix), one obtains:

$$\frac{dC}{d\theta} < 0 \qquad \square$$

θ can therefore be calibrated by a suitable method of search on the basis of the observed mean trip cost.

It is worthwhile to note that there is a great deal of latitude in the choice of travel times and travel costs that are used to calibrate these models. For instance, it may be judged that walking times are far more important in parking lot choice than the in-vehicle travel times. It is straightforward to redefine $c_{\ell,k} = a \cdot c_{ik} + b \cdot c_{kj}$ in the model and determine appropriate values for a and b during the calibration procedure. While the procedure to be used for calibration would involve searching over a range of values for a and b, this generalization does not pose any conceptual difficulties. Similarly, the parking cost may be introduced by using an appropriate conversion factor to generalized cost units.

4. SOLUTION METHOD

The method that we propose to solve both models is the adaptation of the Frank and Wolfe (1956) method for minimizing a convex function subject to linear constraints. The general method is given in Appendix 1 and we only describe below its adaptation for our problems. Other adaptations of the Frank and Wolfe method for similar problems that occur

in the context of transportation planning have been successfully achieved by Nguyen (1976), LeBlanc (1973) and Florian, Nguyen and Ferland (1975).

A typical iteration of the algorithm requires the solution of a linear program that determines the descent direction and the solution of a one dimensional minimization problem of a convex function that determines the step size that achieves the best improvement of the objective function.

We consider first the simpler model (5) - (8). Consider an intermediate stage in the application of the algorithm, when a feasible solution $\bar{f}_{\ell,k}$ is known. The descent direction is obtained by first solving the subproblem:

$$\text{Min} \sum_{\ell} \sum_{k} y_{\ell,k} \{1 + \ell n \, \bar{f}_{\ell,k} + \theta \, c_{\ell,k}\} \tag{46}$$

subject to

$$\sum_{k} y_{\ell,k} = g_{\ell} \quad , \quad \text{all } \ell \tag{47}$$

$$\sum_{\ell} y_{\ell,k} \leq \gamma_k T_k \, , \quad \text{all } k \tag{48}$$

$$y_{\ell,k} \geq 0 \tag{49}$$

This problem is a Hitchcock problem that may be solved efficiently by any of a number of available algorithms (see, for example, Glover, Karney, Klingman and Napier, 1974). The descent

direction is then given by $(\bar{y}_{\ell,k} - \bar{f}_{\ell,k})$ and the optimal step length λ is obtained by solving the problem:

$$\text{Min} \sum_{\ell} \sum_{k} x_{\ell,k} (\ell n \; x_{\ell,k} + \theta \, c_{\ell,k}) \tag{50}$$

where $$x_{\ell,k} = \bar{f}_{\ell,k} + \lambda (\bar{y}_{\ell,k} - \bar{f}_{\ell,k}) \tag{51}$$

Before stating the detailed steps of the algorithm we consider the adaptation of the Frank and Wolfe algorithm for the second model (14) - (18). Given a solution $\bar{f}_{ikj}$, the descent direction is obtained by first solving the sub-problem:

$$\text{Min} \sum_{i} \sum_{k} \sum_{j} y_{ikj} \{1 + \ell n \, \bar{f}_{ikj} + \theta \, c_{ikj}\} \tag{52}$$

subject to $$\sum_{k} \sum_{j} y_{ikj} = O_i \quad , \quad \text{all } i \tag{53}$$

$$\sum_{i} \sum_{k} y_{ikj} = D_j \quad , \quad \text{all } j \tag{54}$$

$$\sum_{i} \sum_{j} y_{ikj} \leq \gamma_k T_k \, , \quad \text{all } k \tag{55}$$

This problem is a linear program that may be solved by any of the packages available such as the MPSX System for IBM Computers. Its solution does not pose any problem, but is likely to be more time consuming than the solution of the sub-problem for the simpler model. The optimal step

length λ is obtained in a similar manner to (50) by setting

$$x_{ikj} = \bar{f}_{ikj} + \lambda (\bar{y}_{ikj} - \bar{f}_{ikj}) \tag{56}$$

and finding λ that minimizes

$$\sum_i \sum_k \sum_j x_{ikj} (\ell n\, x_{ikj} + \theta\, c_{ikj}) \tag{57}$$

In summary, we state the solution algorithms for both problems, by indicating in brackets the necessary steps for the second model.

Step 1 Select an initial solution $\bar{f}_{\ell,k}$ $(\bar{f}_{ikj})$.

Step 2 Determine a descent direction by solving the Hitchcock problem (46) - (49) (the linear program (52) - (55)) , and by setting the direction equal to $\bar{y}_{\ell,k} - \bar{f}_{\ell,k}$ $(\bar{y}_{ikj} - \bar{f}_{ikj})$

Step 3 If $\left| \sum_\ell \sum_k \{1 + \ell n\, \bar{f}_{\ell,k} + \theta\, c_{k,\ell}\} \ (\bar{y}_{\ell,k} - \bar{f}_{\ell,k}) \right| \leq \varepsilon$, for a suitable convergence parameter ε , terminate.

$$\left(\text{if } \left| \sum_i \sum_k \sum_j \{1 + \ell n\, \bar{f}_{ikj} + \theta\, c_{ikj}\} \{\bar{y}_{ikj} - \bar{f}_{ikj}\} \right| \leq \varepsilon\right)$$

Step 4 Solve the one dimensional minimization problem (50) - (51), (56) , (57) and compute the improved solutions

$$\bar{f}_{\ell,k} \leftarrow \bar{f}_{\ell,k} (1 - \lambda) + \lambda\, \bar{y}_{\ell,k}$$

$$(\bar{f}_{ikj} \leftarrow \bar{f}_{ikj} (1 - \lambda) + \lambda\, \bar{y}_{ikj})$$

and return to Step 2.

While we do not give computational results in this paper, it is relevant to comment that there is little doubt that the algorithms are practical for the problems that are encountered in practice, as evidenced by the successful use of similar approaches in solving large scale 'equilibrium' traffic assignment problems such as those reported by Florian and Nguyen (1976).

5. OTHER APPROACHES AND EXTENSIONS

There are other ways to model the choice of parking lots as intermediate destinations. An alternative approach, that has theoretical appeal, is to include the parking choice in the route choice from origin to destination and not treat it separately. In urban road networks that are subject to congestion, the assumption that route choices are taken according to Wardrop's (1952) second principle (user-optimization) has become well accepted as a reasonable representation of actual choices made. See for instance the validation study of Florian and Nguyen (1976). The adaptation of this principle to the problem we treat in this paper implies that, between an origin and a destination, trip makers use composite paths (from origin to parking lot to destination) of equal (generalized) cost. If we assume, such as for the first model that we have presented, that an origin destination matrix g_{ij} is known, the problem becomes an equilibrium traffic assignment problem on a network. The flow of vehicular traffic through parking lots must be modelled explicitly on this network such as indicated in Figure 1.

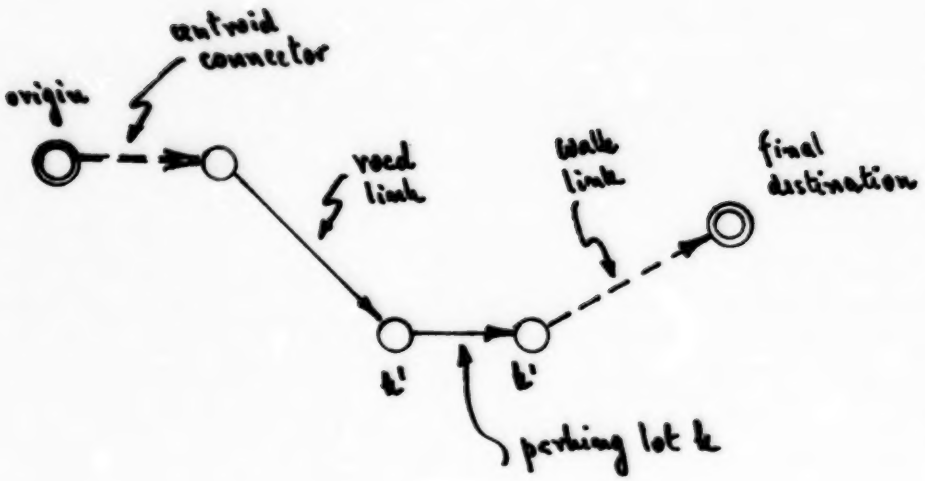

Figure 1

The network is composed of centroid connectors, road links, parking lot links and walk links. The travel cost associated with each road link is given by convex increasing functions of the link volumes, the travel cost associated with each walk link is the perceived cost of the walking time and distance and the travel cost function associated with each parking lot link (k,k') represent the parking cost, parking time and its capacity. The latter cost function may be of the general form

$$C_{k,k'}(v_{k,k'}) = a / (v_{k,k'} - T_{k,k'})^{b}$$

where $C_{k,k'}$ is the cost at flow level $v_{k,k'}$, $T_{k,k'}$ is the capacity of lot k, and a,b are constants that may be calibrated by using survey data.

A difficulty associated with this approach is that the flow on the road links is vehicular, while on walk links the flow is that of pedestrians. Thus the flow on the links (k,k') must be multiplied by the car occupancy factor γ_k ; the resulting problem is a network

flow problem with gains (see for instance Jewell, 1962). Thus the standard equilibrium traffic assignment codes such as TRAFFIC (Nguyen and James, 1975) or the UROAD module in the UTPS package (U.S. Department of Transportation, 1976), may not be used without making some nontrivial modifications. In addition, we believe that the behavioural assumption made in such an approach is too simplistic and we doubt that it would capture the variety of behavioral causes that affect parking lot choice. The models that we proposed earlier, while also rather simple, are likely to be better calibrated against observed behaviour.

We remark that if the approach presented above is used with the assumption that only the number of person trips originating at i, O_i and the number of person trips destined for j, D_j are known, such as we have done for the second model, the corresponding model would be a combined distribution-assignment problem such as studied by Florian, Nguyen and Ferland (1975). We have the same reservations about the use of this approach as above. The advantage of this approach is that the road congestion and parking congestion is explicitly taken into account, however the basic algorithm must be modified to handle flows with gains and it is questionable whether it is behaviourally valid to extend Wardrop's principle of route choice to include parking allocation.

The models that we have proposed in this paper may be extended to situations other than downtown parking, whenever there is a mode transfer and the possibility of parking at the transfer point, such as the familiar 'park-and-ride' situations in suburban areas. An attempt to model station choice, and therefore parking lot choice, is due to G. Desfor (1973) in the context of the Lindenwold high-speed line. He considers the choice

between two stations, one final destination (all the CBD) and does not take into account explicitly the parking capacity constraints, although he found that the station choice is quite sensitive to the availability of parking space. The models that we suggest can take all these factors into account.

Another possible extension is to consider two modes of access to the parking lot, such as getting a ride to the station ('kiss-and-ride' and 'park-and-ride'). This would involve more complicated entropy maximization formulations where the variables $f^m_{\ell,k}$ denotes the number of trips taken by mode m from origin-destination pair ℓ through parking lot k.

6. CONCLUSION

The main contribution of this paper has been to show, in the context of modelling parking lot choice, that it is possible to extend entropy maximization trip distribution type models to take into account inequality constraints as well as equality constraints. Such models may be easily calibrated by using generalization of well known calibration methods. Their use, in the conventional sequence of the urban transportation planning process may follow the trip generation stage (second model) or the trip distribution stage (first model). The models that we propose are more satisfactory than the linear programming models for parking lot allocation since they are more general and may be calibrated, thus excluding extremal solutions that may be a poor representation of

the demand for parking.

We believe that there are many modelling challenges that remain to be studied in the context of parking lot choice. One of those is the possibility of introducing parking lot capacities in models that treat parking lot choice in a more refined behavioral way.

7. REFERENCES

AUSTIN, T.W. (1973). Allocation of parking demand in a CBD. Highway Research Record #444, 1-8.

DESFOR, G.L. (1973). Binary station choice models: An analysis of the market areas for the suburban stations on the Lindenwold high-speed line. Ph.D. dissertation in City and Regional Planning, University of Pennsylvania.

EVANS, A.W. (1971). The calibration of trip distribution models with exponential or similar cost functions. Transportation Research 5, 15-38.

FELLER, W. (1968). An Introduction to Probability Theory and its Applications. John Wiley.

FLORIAN, M. and NGUYEN, S. (1976). An application and validation of equilibrium traffic assignment methods. Transportation Science 10.

FLORIAN, M., NGUYEN, S. and FERLAND, J. (1975). On the combined distribution assignment of traffic. Transportation Science 9, 43-53.

FRANK, M. and WOLFE, P. (1956). An algorithm for quadratic programming. Naval Research Logistics Quarterly 3, 95-110.

GILLEN, D.W. (1975). Effects of changes in parking prices and parking restrictions in urban transport demands and congestion levels. Research Report # 25, University of Toronto-York University Joint Program in Transportation.

GLOVER, F., KARNEY, D., KLINGMAN, D. and NAPIER, A. (1974). A computational study on start procedures, basis change criteria and solution algorithms for transportation problems. Management Science 20, 793-814.

JEWELL, W.S. (1962). Optimal flow through networks with gains. Operations Research 10, 476-499.

KADAS, Sandor A. and KLAFSZKY, Emil (1976). Estimation of the parameters in the gravity model for trip distribution: A new method and solution algorithm. Regional Science and Urban Economics 6, No. 4, 439-457.

KANAFANI, A.K. (1972). Location model for parking facilities. Transportation Eng. Jour. ASCE 98, 117-129.

LEBLANC, L. (1973). Mathematical programming algorithms for large scale network equilibrium and network design problems. Ph.D. dissertation, Department of Industrial Engineering and Management Sciences, Northwestern University.

NGUYEN, S. (1976). A unified approach to equilibrium methods of traffic assignment. In Traffic Equilibrium Methods, Proceedings 1974, Vol. 118, Lecture Notes in Economics and Mathematical Systems, Springer Verlag, New York, 148-179.

NGUYEN, S. and JAMES, L. (1975). TRAFFIC - An equilibrium traffic assignment program. Publication # 17, Centre de recherche sur les transports, Université de Montréal.

UMTA Transportation Planning System (UTPS) User's Guide (1976). Washington, D.C., U.S. Department of Transportation, Urban Mass Transportation Administration, Office of Transit Planning, Planning Methodology and Technical Support Division.

WARDROP, J.G. (1952). Some theoretical aspects of road traffic research. Proc. Inst. Civil Engineers Part II, 1, 325-378.

WESTIN, R.B. and GILLEN, D.W. (1976). Parking policies and transit demand: A case study of endogenous attributes in disaggregate mode choice models.

WHITLOCK, E.M. (1973). Use of linear programming to evaluate alternative parking sites. Highway Research Record #444, 9-19.

WILLIAMS, I. (1976). A comparison of some calibration techniques for doubly constrained models with an exponential cost function. Transportation Research 10, 91-104.

WILSON, A.G. (1970). Entropy in Urban and Regional Modelling, Pion, London.

ZANGWILL, W.I. (1969). Nonlinear Programming: A Unified Approach. Englewood Cliffs, N.J.: Prentice Hall.

APPENDIX 1

The Frank and Wolfe algorithm [8]

Consider the problem of minimizing a convex function subject to linear constraints:

$$\text{Min } Z(x) \text{ , subject to } A x \leq b \text{ , } X \geq 0 .$$

The detailed steps of the Frank and Wolfe algorithm for solving this problem are:

Step 1 Given a feasible solution X^1 , set $\ell = 1$.

Step 2 Determine Y^ℓ that minimizes

$$\nabla Z(X^\ell)\, Y \text{ , subject to } A Y \leq b \text{ , } Y \geq 0 .$$

Step 3 Set the descent direction $d^\ell = Y^\ell - X^\ell$.

If $| \nabla Z(X^\ell)\, d^\ell | \leq \varepsilon$, terminate, where ε is a suitable convergence parameter. (X^ℓ is the optimal solution)

Step 4 Find the optimal step length λ^ℓ that minimizes

$$Z(X^\ell + \lambda\, d^\ell) \quad \text{for } 0 \leq \lambda \leq 1 .$$

Step 5 Revise the current solution

$$X^{\ell+1} = X^\ell + \lambda^\ell\, d^\ell ;$$

set $\ell = \ell + 1$ and return to Step 2.